

A Simple Proof of the Chiral Gravity Conjecture

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Abstract

Chiral gravity is three-dimensional asymptotically AdS_3 gravity with an Einstein, cosmological, and Chern-Simons term at a critical value of the coupling denoted $\mu\ell = 1$. Ordinarily, excitations of AdS_3 gravity are known to transform non-trivially under an asymptotic symmetry group consisting of both a left-moving and a right-moving conformal group. However it was recently conjectured [1] that at the chiral point $\mu\ell = 1$ all excitations are chiral in that they transform only under the right-moving conformal group. We show herein that at the chiral point, the group of trivial diffeomorphisms is enhanced to include the left-moving conformal transformations, and the asymptotic symmetry group contains only one (right-moving) copy of the conformal group. Diffeomorphism invariance then requires that all physical excitations are annihilated by left-moving conformal transformations, establishing nonperturbatively the chiral nature of chiral gravity.

Topologically massive gravity (TMG)[2, 3] with a negative cosmological constant is described by the action

$$I_{TMG} = \frac{1}{16\pi G} \left[\int d^3x \sqrt{-g} (R + 2/\ell^2) + \frac{1}{\mu} I_{CS} \right] \quad (1)$$

where I_{CS} is the gravitational Chern-Simons action

$$I_{CS} = \frac{1}{2\pi} \int_{\mathcal{M}} d^3x \sqrt{-g} \varepsilon^{\lambda\mu\nu} \Gamma_{\lambda\sigma}^r \left(\partial_\mu \Gamma_{r\nu}^\sigma + \frac{2}{3} \Gamma_{\mu\tau}^\sigma \Gamma_{\nu r}^\tau \right) \quad (2)$$

and G has the conventional positive sign. Chiral gravity [1] is defined by taking $\mu\ell \rightarrow 1$ and imposing the standard Brown-Henneaux [4] asymptotically AdS_3 boundary conditions. As these are local we can describe them in Poincare coordinates for AdS_3

$$ds^2 = \ell^2 \left(\frac{dx^+ dx^- + dy^2}{y^2} \right) \quad (3)$$

with the boundary at $y = 0$. In these coordinates, the Brown-Henneaux boundary conditions are that fluctuations $h_{\mu\nu}$ of the metric about (3) fall off at the boundary according to

$$\begin{pmatrix} h_{++} = \mathcal{O}(y^0) & h_{+-} = \mathcal{O}(y^0) & h_{+y} = \mathcal{O}(y) \\ h_{-+} = h_{+-} & h_{--} = \mathcal{O}(y^0) & h_{-y} = \mathcal{O}(y) \\ h_{y+} = h_{+y} & h_{y-} = h_{-y} & h_{yy} = \mathcal{O}(y^0) \end{pmatrix} \quad (4)$$

The most general diffeomorphism which preserves (4) is of the form

$$\zeta = \zeta^+ \partial_+ + \zeta^- \partial_- + \zeta^y \partial_y \quad (5)$$

$$= \left[\epsilon^+(x^+) + \frac{y^2}{2} \partial_-^2 \epsilon^-(x^-) + \mathcal{O}(y^4) \right] \partial_+ \quad (6)$$

$$+ \left[\epsilon^-(x^-) + \frac{y^2}{2} \partial_+^2 \epsilon^+(x^+) + \mathcal{O}(y^4) \right] \partial_- \quad (7)$$

$$+ \left[y \partial_+ \epsilon^+(x^+) + y \partial_- \epsilon^-(x^-) + \mathcal{O}(y^3) \right] \partial_y. \quad (8)$$

These are parameterized by a left and a right moving function $\epsilon^+(x^+)$ and $\epsilon^-(x^-)$. The subleading terms all correspond to trivial diffeomorphisms, i.e. they have no finite surface term and hence vanish when the constraints are imposed. The asymptotic symmetry group (ASG) is defined as the general boundary-condition-preserving diffeomorphism (5) modulo the trivial diffeomorphisms. For generic μ the ASG is generated by two copies of the Virasoro algebra. We shall see the case $\mu\ell = 1$ requires special attention.

The consistency of these boundary conditions for generic μ was demonstrated in [5]. This demonstration involves in particular showing that *all* the generators (5) which preserve the asymptotically AdS_3 boundary conditions are well-defined expressions free of singularities as $y \rightarrow 0$ for *any* metric of the form (4). The generator of a diffeomorphism ζ is

$$Q[\zeta] = \int_{\partial\Sigma} \sqrt{\sigma} u^i T_{ij} \zeta^j, \quad (9)$$

where the integral is over the S^1 boundary of a spatial slice Σ , σ is the induced metric on $\partial\Sigma$, u^i is the timelike unit normal to Σ , and T_{ij} is the boundary stress tensor. Under Dirac brackets, the generators associated with asymptotic symmetries obey the same algebra as the symmetries themselves, up to a possible central term. The boundary stress tensor appearing in (9) is modified from that originally given for Einstein gravity in [4] due to the Chern-Simons term [6, 7, 8, 9]. Explicit expressions for the generators can be found in different forms in various places, the most complete discussion being [5]. We will work in an asymptotic gauge $h_{+-} = \mathcal{O}(y)$, which can always be reached with a trivial diffeomorphism of the form $\zeta = f(x^+, x^-) y^3 \partial_y$. An especially convenient expression for the boundary stress tensor found in reference [7] then reduces to

$$T_{++} = \left(1 + \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} h_{++}, \quad (10)$$

$$T_{--} = \left(1 - \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} h_{--}, \quad (11)$$

$$T_{+-} = 0. \quad (12)$$

$$(13)$$

The generators of the ASG are then

$$Q(\zeta) = \left(1 + \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} \int_{\partial\Sigma} dx^+ h_{++} \epsilon^+ + \left(1 - \frac{1}{\mu\ell}\right) \frac{1}{8\pi G\ell} \int_{\partial\Sigma} dx^- h_{--} \epsilon^-, \quad (14)$$

For chiral gravity at $\mu\ell = 1$, this further reduces to the simple expression

$$Q(\zeta) = \frac{1}{4\pi G\ell} \int_{\partial\Sigma} dx^+ h_{++} \epsilon^+. \quad (15)$$

As $\epsilon^-(x^-)$ does not appear in this expression, all the diffeomorphisms parameterized by $\epsilon^-(x^-)$ are trivial, including the global $SL(2, R)_L$ subgroup. Hence the ASG is reduced to one right-moving copy of the conformal group. Gauge invariance requires

nonpertubatively that all physical excitations must be annihilated by $Q(\epsilon^-)$ and hence may carry only right moving quantum numbers. This is of course consistent with the observation of [1] that the left-moving energy of massless gravitons, massive gravitons¹ and BTZ black holes all vanish as $\mu\ell \rightarrow 1$.

With hindsight this result could be anticipated simply from the fact that $c_L = 0$ for chiral gravity. If the central charge is nonzero, the associated Virasoro generators cannot be trivial, but this obstruction vanishes for the left moving sector of chiral gravity since $c_L = \frac{3\ell}{2G}(1 - \frac{1}{\mu\ell})$. It further follows from the Virasoro algebra with $c_L = 0$ that the left-moving boundary gravitons, which are Virasoro descendents of the AdS_3 vacuum, have zero norm. Hence they should be trivially pure gauge as we have indeed seen explicitly above.

We conclude with several comments:

- (i) We wish to emphasize that while we have established nonperturbatively the chirality of chiral gravity, we have not proven that the spectrum of right-moving energies is positive or bounded from below. This remains an outstanding challenge central to the question of vacuum stability.
- (ii) It is possible that at the chiral point (4) are not the only consistent boundary conditions. For pure Einstein gravity, (4) are known to be the only consistent possibility [4]², and from the work of [5] this seems likely to be the only consistent possibility for generic μ . However at the chiral point, since fewer components of the metric appear in the generators, it might be possible to weaken the boundary conditions and thereby get a larger ASG (e.g. an additional current algebra).³ Another interesting and more general possibility, not considered here, is "asymptotically non-linear" boundary conditions [12, 13, 14], in which the allowed symmetry generators are taken to depend on the asymptotic values of the fields.
- (iii) At the chiral point, the trivial symmetry group is enlarged to include the left-moving conformal transformations. At the same time, all left-moving degrees of freedom become trivially pure gauge. This phenomenon of losing degrees of freedom due to an enhanced gauge symmetry appeared in a related AdS context in [15].
- (iv) Following the conjecture [1] on the chiral nature of chiral gravity, several purported counterexamples have appeared [16, 18, 19, 20, 21, 22, 23]. None of them take

¹We note that the quantity denoted h_L in that paper is related to the physical left moving energy as given in [10, 11, 7] by a factor involving $(1 - 1/\mu\ell)$.

²Except of course for the trivial case when the boundary conditions are so strict as to forbid all excitations [4].

³The proposal of [18] that the boundary conditions for h_{++} , h_{--} and h_{+-} be weakened from $\mathcal{O}(y^0)$ to $\mathcal{O}(\ln y)$ is inconsistent in this sense because the generators (15) are ill-defined. Another possibility not so obviously ruled out is to take only h_{--} to be $\mathcal{O}(\ln y)$.

into account the enhanced symmetry at the chiral point; and must all fail to be true counterexamples to the chirality conjecture in one way or another. For example in [16] linearized eigensolutions to TMG were analyzed in light cone gauge and Poincare coordinates. These eigensolutions are all singular on the global AdS_3 boundary[17], but it was nevertheless argued that by formation of wavepackets and a singular gauge transformation, smooth finite energy solutions with the asymptotics (4) could be constructed. In general it is extremely difficult to analyze global properties of AdS_3 using both a gauge and a coordinate system which are singular on the boundary. In [18] a non-chiral linearized solution was explicitly found in global coordinates, but it does not obey the required boundary conditions (4) for chiral gravity. The recent paper [19] claims to find a related family of nonchiral solutions which do obey (4), but as $SL(2, R)_L$ acts trivially on the physical solution space many of these are trivial gauge transformations of one another.⁴ Other constructions of purported non-chiral modes do not fully address global issues [20, 21, 22, 23].

Acknowledgements

This work was partially funded by DOE grant DE-FG02-91ER40654. I wish to thank Tom Hartman, Wei Li, Alex Maloney and Wei Song for useful conversations.

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⁴This does not preclude the possibility that these modes have an interesting chiral right-moving component.

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